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Research Article

Transformation of Surface Charge Density in Mixed Number Lorentz Transformation

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Abstract

We know that the electric charge of an isolated system is relativistically invariant. We have studied the transformation of surface charge density in Special, Most general, Mixed number Lorentz transformation. As the formula of length contraction is not the same in these types of Lorentz transformation, the transformation equation of surface charge density will be different in the above mentioned transformations.

Keywords: Special Lorentz Transformation; Surface charge density

1. INTRODUCTION

In most treatments on special relativity[1], the line of motion is aligned with the *x*-axis. In such a situation the (y,z) coordinates are invariant under the Lorentz transformations. However, it is of interest to study the case when the line of motion does not coincide with any of the coordinate axes. Practical instances of such a situation are an airplane during landing or take off. The ground at the air field has a natural coordinate system with the *x*-axis parallel to the ground, whereas the airplane ascends or descends at an angle with the ground. In this paper we derive the transformation equations for surface charge density in different types of Lorentz transformation when the line of motion is not aligned with any of the coordinate axes.

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1.1 Special Lorentz transformation

Let us consider two inertial frames of reference S and S´, where the frame S is at rest and the frame S´ moving along the X axis with velocity v with respect to the S frame. The space and time coordinates of S and S´ are (x, y, z, t) and (x´, y´, z´, t´) respectively. The relation between then coordinates of S and S´, which is called the special Lorentz transformation, can be written as[1]

$$x' = \gamma(x - vt), \tag{1a}$$
$$y' = y, \tag{1b}$$
$$z' = z, \tag{1c}$$
$$t' = \gamma(t - vx), \tag{1d}$$

where $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$ and c = 1.

And

$$x = \gamma(x' + vt') \tag{2a}$$
$$y = y', \tag{2b}$$
$$z = z', \tag{2c}$$
$$t = \gamma(t' + vx') \tag{2d}$$

1.2 Most general Lorentz transformation

When the velocity $\boldsymbol{v}$ of S´ with respect to S is not along the x-axis, i.e. the velocity $\boldsymbol{v}$ has three components v_x, v_y and v_z, then the relation between the coordinates of S and S´, which is called the Most General Lorentz transformation, can be written as[2]

$$\mathbf{x}' = \mathbf{x} + \mathbf{v}\left[\left\{(\mathbf{x}.\mathbf{v})/v^2\right\}(\gamma - 1) - t\gamma\right]$$
$$t' = \gamma(t - \mathbf{x}.\mathbf{v})$$

Where, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$ and c = 1

The inverse most general Lorentz transformation can be written as

$$\mathbf{x} = \mathbf{x}' + \mathbf{v}\left[\left\{(\mathbf{x}'.\mathbf{v})/v^2\right\}(\gamma - 1) + t'\gamma\right]$$
$$t = \gamma(t' + \mathbf{x}'.\mathbf{v}) \tag{3}$$

1.3 Mixed-number Lorentz transformation

Mixed-number Lorentz transformation can be generated using Mixed number algebra[5-9, 13]. Let us consider the velocity $\boldsymbol{v}$ of the S´ frame with respect to the S frame is not along x-axis i.e. the velocity $\boldsymbol{v}$ has three components v_x, v_y, v_z. Let in this case $\boldsymbol{z}$ and $\boldsymbol{z}'$ be the space part in S and S´ frame respectively. In this case using equations (1a) and (1d), we can write

$$(t'+\mathbf{z}') = \gamma\{(t+\mathbf{z}) - (t+\mathbf{z})\mathbf{v}\}$$

or, $$(t'+\mathbf{z}') = \gamma\{(t+\mathbf{z}) - (t+\mathbf{z})(0+\mathbf{v})\} \tag{4}$$

Using the product rule of two mixed number algebra[6-8], we can write

$$(t+\mathbf{z})(0+\mathbf{v}) = \mathbf{z.v} + t\mathbf{v} + i\mathbf{z}\times\mathbf{v}$$

Putting this in Eq. (4), we get

$$(t'+\mathbf{z}') = \gamma\{(t+\mathbf{z}) - (\mathbf{z.v} + t\mathbf{v} + i\mathbf{z}\times\mathbf{v})\}$$

or, $$(t'+\mathbf{z}') = \gamma(t - \mathbf{z.v}) + \gamma(\mathbf{z} - t\mathbf{v} - i\mathbf{z}\times\mathbf{v}) \tag{5}$$

The left hand side of equation (5) is the sum of a scalar t' and a vector $\boldsymbol{z}'$ and the right hand side is also the sum of a scalar $\gamma(t\text{-}\boldsymbol{z.v})$ [4] and a vector $\gamma(\boldsymbol{z}\text{-}t\boldsymbol{v}\text{-}i\boldsymbol{z}\times\boldsymbol{v})$. So according to Mixed number algebra, equating the scalar and vector parts of Eq. (5) we can write:

$$\mathbf{z}' = \gamma(\mathbf{z} - t\mathbf{v} - i\mathbf{z}\times\mathbf{v}) \tag{6a}$$

$$t' = \gamma(t - \mathbf{z.v}) \tag{6b}$$

Similarly it can be shown that

$$\mathbf{z} = \gamma(\mathbf{z}' + t'\mathbf{v} + i\mathbf{z}'\times\mathbf{v}) \tag{7a}$$

And

$$t = \gamma(t' + \mathbf{z}'.\mathbf{v}) \tag{7b}$$

Equations (6a), (6b), (7a) and (7b) are the Mixed number Lorentz transformation[5-7, 13].

2. SURFACE CHARGE DENSITY

The surface charge density is the amount of electric charge in a unit surface[3]. In other words the amount of electric charge per unit surface area is called surface charge density[3]. It is measured in coulombs per square meter *(C/m²)*. Since there are positive as well as negative charges, the charge density can take on negative values. Like any density it can depend on position. We know that the charge on the electron or proton is the minimum, called the elementary charge $e\ (= 1.6\times10^{-19} coul.)$. The electric charge is discrete which may be determined by counting the number of elementary charged particles. As the total number of elementary charges cannot depend on the state of the motion of the observer, we may conclude that the electric charge is relativistically invariant. Based on this important conclusion we want to calculate the transformation equation for the charge density σ.

3. TRANSFORMATION OF SURFACE CHARGE DENSITY

3.1 Transformation of Surface Charge Density using Special Lorentz transformation

Let us consider two systems S and S', S' is moving with uniform velocity ***v*** relative to S along negative direction of *X*-axis as shown in Fig. 1. Let there be a stationary sheet of uniform charge density $+\sigma$ coul/m^2 at rest in system S having one edge parallel to *X*'-axis. Let the sheet be a square of side L_0 and placed parallel to *X-Y* plane. The observer in the system S' will observe that the sheet is moving with velocity ***v*** along (+) ve *X*-axis.

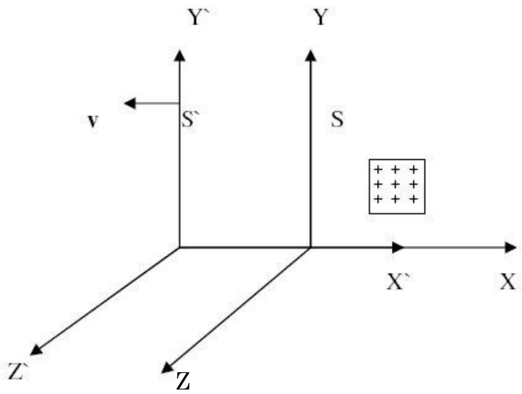

Figure 1: The system S' is moving along *X*-axis with velocity ***v*** relative to the system S.

We have the transformation relation of surface charge density from the principle of invariance of charge which states, ‘The total electric charge in an isolated system is relativistically invariant’.

The total charge as observed in the system S is

$$Q_0 = L_0{}^2\sigma \tag{8}$$

The observer in system S' will noticed that the side of the square along *X*-axis has been contracted from L_0 to L, where

$$L = \frac{L_0}{\gamma} = \left(\sqrt{1-\frac{v^2}{c^2}}\right)L_0$$

As the length is contracted only along the *X*-axis, the charge as observed by observer in the system S' is

$$Q' = \left(\sqrt{1-\frac{v^2}{c^2}}\right)L_0{}^2\sigma' \tag{9}$$

Where σ' is the surface charge density in system S'. According to the principle of conservation of charge

$$Q' = Q_0$$

or, $$\left(\sqrt{1-\frac{v^2}{c^2}}\right)L_0{}^2\sigma' = L_0{}^2\sigma$$ [Using Eqs. (8) and (9)]

$$\therefore \sigma' = \frac{\sigma}{\sqrt{1-v^2/c^2}} = \gamma\sigma$$

This equation represents the transformation equation for the surface charge density[3] in special Lorentz transformation (Table 1).

3.2 Transformation of surface Charge Density using Most General Lorentz transformation

Let us consider two systems S and S' where the frame S' is moving with velocity ***v*** relative to the system S along *X-Y* plane as shown in Fig. 2. Thus the velocity of ***v*** has two components v_x and v_y. Let us consider a stationary sheet of uniform charge density $+\sigma$ *coul/m*^{2} at rest in system S having one edge parallel to *X*-axis and let the sheet be a square of side L_0 placed parallel to *X-Y* plane. The observer in the system S' will observe that the sheet is moving in opposite direction.

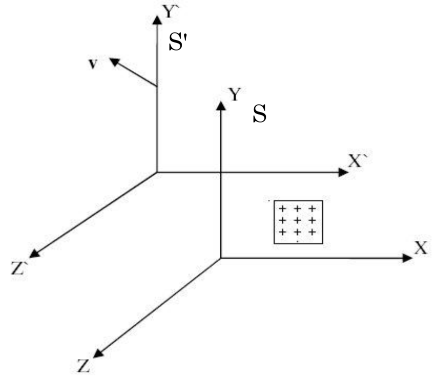

Figure 2: The system S' is moving in *X-Y* plane with velocity ***v*** relative to the system S.

If L_0 is the length of the square charged sheet in S, then the length contraction in the moving frame for the most general Lorentz transformation can be written as[12]

$$\mathbf{L}_0 = \mathbf{L} + \frac{\mathbf{v}}{v^2}\left[x_2' v\cos\theta - x_1' v\cos\theta\right](\gamma - 1)$$

or, $$\mathbf{L}_0 = \mathbf{L} + \frac{\mathbf{v}}{v} L\cos\theta(\gamma - 1)$$

or, $$L_0{}^2 = \left[\mathbf{L} + \frac{\mathbf{v}}{v} L\cos\theta(\gamma - 1)\right].\left[\mathbf{L} + \frac{\mathbf{v}}{v} L\cos\theta(\gamma - 1)\right]$$

or, $$L_0{}^2 = L^2\left[1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2\right]$$

$$\therefore L^2 = \frac{L_0{}^2}{1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2}$$

or, $$L_x^{\ 2} + L_y^{\ 2} = \frac{L_{0_x}^{\ 2} + L_{0_y}^{\ 2}}{1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2}$$

$$\therefore L_x^{\ 2} = \frac{L_{0_x}^{\ 2}}{1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2}$$

and $$L_y^{\ 2} = \frac{L_{0_y}^{\ 2}}{1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2}$$

The total charge observed by an observer in S' is

$$Q' = L_x L_y \sigma'$$

$$= \sqrt{\frac{L_{0_x}^{\ 2}}{1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2}} \cdot \sqrt{\frac{L_{0_y}^{\ 2}}{1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2}}\,\sigma'$$

$$= \frac{L_0^{\ 2}}{1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2}\,\sigma' \quad \text{[for square charge sheet]} \tag{10}$$

Where σ' is the surface charge density in system S'.

According to the principle of conservation of charge

$$Q' = Q_0$$

Using Eqs. (8) and (10), this can be written as

$$\frac{L_0^{\ 2}}{1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2}\,\sigma' = L_0^{\ 2}\sigma$$

or, $$\sigma' = \left[1 + 2\cos^2\theta(\gamma - 1) + \cos^2\theta(\gamma - 1)^2\right]\sigma$$

This equation represents the transformation equation for surface charge density when the system S' is moving in *X*-*Y* plane (Table 1).

3.3 Transformation of Surface Charge Density with Mixed number Lorentz transformation

Let us consider two systems S and S' where the system S' is moving with the velocity ***v*** relative to the system S along any arbitrary direction as shown in the figure-2 so that the velocity of ***v*** have two components v_x and v_y.

Let us consider a stationary sheet of uniform charge density $+\sigma$ *coul/m*^{2} at rest in system S having one edge parallel to *X*-axis and let the sheet be a square of side L_0 placed parallel to *X*-*Y* plane. The observer in the system S' will observe that the sheet is moving in opposite direction with velocity ***v*** along any arbitrary direction as shown in the figure 2.

If L_0 is the length of the square charged sheet in S, then the length contraction in the moving frame for the mixed number Lorentz transformation can be written as[12]

$$\mathbf{L}_0 = \gamma(\mathbf{L} - i\mathbf{L}\times\mathbf{v})$$

or, $$L_0^{\ 2} = \gamma^2[L^2 - \mathbf{L}.i(\mathbf{L}\times\mathbf{v}) - i(\mathbf{L}\times\mathbf{v}).\mathbf{L} + i(\mathbf{L}\times\mathbf{v}).i(\mathbf{L}\times\mathbf{v})$$

or, $$L_0^{\ 2} = \gamma^2[L^2 + i^2(L^2v^2 - L^2v^2\cos^2\theta)]$$

$$= \gamma^2 L^2[1 - v^2(1-\cos^2\theta)]$$

$$\therefore L^2 = \frac{L_0^{\ 2}}{\gamma^2[1-v^2(1-\cos^2\theta)]}$$

or, $$L_x^{\ 2} + L_y^{\ 2} = \frac{L_{0x}^{\ 2} + L_{0y}^{\ 2}}{\gamma^2[1-v^2(1-\cos^2\theta)]}$$

$$\therefore L_x^{\ 2} = \frac{L_{0x}^{\ 2}}{\gamma^2[1-v^2(1-\cos^2\theta)]} \text{ and } L_y^{\ 2} = \frac{L_{0y}^{\ 2}}{\gamma^2[1-v^2(1-\cos^2\theta)]}$$

The total charge as observed by an observer in S' is

$$Q' = L_x L_y \sigma'$$

$$= \sqrt{\frac{L_{0x}^{\ 2}}{\gamma^2[1-v^2(1-\cos^2\theta)]}}.\sqrt{\frac{L_{0y}^{\ 2}}{\gamma^2[1-v^2(1-\cos^2\theta)]}}\sigma'$$

$$= \frac{L_0^{\ 2}}{\gamma^2[1-v^2(1-\cos^2\theta)]}\sigma' \qquad \text{[for square charge sheet]} \qquad (11)$$

Where σ' is the surface charge density in system S'.

According to the principle of conservation of charge

$$Q' = Q_0$$

or, $$\frac{L_0^{\ 2}}{\gamma^2[1-v^2(1-\cos^2\theta)]}\sigma' = L_0^{\ 2}\sigma \quad \text{[using Eqs. (8) and (11)]}$$

or, $$\sigma' = \gamma^2[1-v^2(1-\cos^2\theta)]\sigma$$

This equation represents the transformation equation for the surface charge density in Mixed number Lorentz transformation (Table1).

Table 1: Transformation equations for surface charge density in different types of Lorentz transformation (LT).

	Special LT	Most general LT	Mixed-number LT
Space	$x' = \gamma\,(x - vt)$, $y' = y$, $z' = z$,	$x' = \{1 + (\gamma-1)v_x{}^2/v^2\}x + (\gamma-1)v_x v_y y/v^2$ $+ (\gamma-1)v_x v_z z/v^2 - v_x\gamma t$ $y' = (\gamma-1)v_y v_x x/v^2 + \{1 + (\gamma-1)v_y{}^2/v^2\}y$ $+ (\gamma-1)v_y v_z z/v^2 - v_y\gamma t$ $z' = (\gamma-1)v_z v_x x/v^2 + (\gamma-1)v_y v_z y/v^2$ $+ \{1 + (\gamma-1)v_z{}^2/v^2\}z - v_z\gamma t$	$x' = \gamma x - t\gamma v_x - i\gamma(yv_z - zv_y)$ $y' = \gamma y - t\gamma v_y - i\gamma(zv_x - xv_z)$ $z' = \gamma z - t\gamma v_z - i\gamma(xv_y - yv_x)$
Time	$t' = \gamma(t - vx)$	$t' = -\gamma v_x x - \gamma v_y y - \gamma v_z z + \gamma t$	$t' = \gamma(t - xv_x - yv_y - zv_z)$
Length contraction	$L_0 = \gamma L$	$\mathbf{L}_0 = \mathbf{L} + \frac{\mathbf{v}}{v} L\cos\theta(\gamma-1)$	$\mathbf{L}_0 = \gamma(\mathbf{L} - i\mathbf{L}\times\mathbf{v})$
Surface charge density	$\sigma' = \gamma\sigma$	$\sigma' = [1 + 2\cos^2\theta(\gamma-1) + \cos^2\theta(\gamma-1$	$\sigma' = \gamma^2[1 - v^2(1-\cos^2\theta)]\sigma$

Table 2: Numerical values of surface charge densities of moving system in terms of that of rest system in Different types of Lorentz transformation (LT)

Parameters		Special LT	Most general LT	Mixed-number LT
$\theta = 0^0$	v = .5c	1.155 σ	-	-
	v = .7c	1.4 σ	-	-
$\theta = 30^0$	v = .5c	-	1.25 σ	1.25 σ
	v = .7c	-	1.73 σ	1.73 σ
$\theta = 45^0$	v = .5c	-	1.16 σ	1.16 σ
	v = .7c	-	1.48 σ	1.48 σ
$\theta = 60^0$	v = .5c	-	1.08 σ	1.08 σ
	v = .7c	-	1.24 σ	1.24 σ

* calculations were carried out taking c as unity.

4. CONCLUSION

The transformation formula for surface charge density in Special, Most general, Mixed number Lorentz transformation are illustrated in Table 1. It has been observed that the formula of the surface charge density in case of most general Lorentz transformation is more complicated than mixed number Lorentz transformation. The numerical values of surface charge densities of the moving system in terms of a system at rest for different types of Lorentz transformation were calculated as shown in Table 2. From Table 2 it is observed that for the same angle (θ) between the systems S and S′, the value of surface charge density of moving system (σ'), increases with increasing velocity ***v*** of the moving system and for the same velocity of the moving system σ' decreases with increasing the angle θ. Although the results obtained in case of most general Lorentz transformation and mixed number Lorentz transformation are the same, the calculation is easier in case of mixed number Lorentz transformation. So, it can be convenient to use mixed number Lorentz transformation when the line of motion does not coincide with any of the coordinate axes.

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