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Research Article

Approximate *l*-States Solutions to the Schrodinger Equation with Manning-Rosen plus Hellmann Potential via WKB Approximation Scheme

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Abstract

The approximate analytical solutions of the radial Schrodinger equation have been obtained by the interaction of Manning-Rosen and Hellmann potentials which is a newly proposed potential. Using the Wentzel-Kramers-Brillouin WKB approach, we obtained the eigenstates solutions for any arbitrary angular momentum. Special cases of potential consideration have been discussed. Eigenenergy solutions to equations obtained play an important role in quantum mechanics because they contain a wealth of vital information regarding the system under consideration.

Keywords: Schrodinger equation; Manning-Rosen potential; Hellmann potential; WKB approximation.

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1. INTRODUCTION

In quantum mechanics, exact solutions to equations play an important role because they contain a wealth of important information regarding the system under consideration. For example, the exact solution of the Schrodinger equation for the hydrogen atom and simple harmonic oscillator provided strong evidence supporting the validity of the quantum theory. However many quantum systems are treated as approximations because exact solutions are few[1-4]. The bound state energy equation and the unnormalized radial wave functions have been approximately obtained for the Manning-Rosen potential by using the supersymmetric WKB approach and the function analysis method[5]. The analytical bound state solutions of the Dirac equation with the Manning–Rosen potential for an arbitrary spin-orbit coupling quantum have been solved[6].

The WKB approximation method is one of the earliest and simplest methods of obtaining approximate eigenvalues of the one-dimensional Schrodinger equation in the limiting case of large quantum numbers and was originally proposed Wentzel, Kramers, and Brillouin[7-10]. In the lowest- order approximation, the WKB quantization condition is

$$\int_{r_2}^{r_1} \sqrt{2m\,(E - V(r)}\,dr = \pi\hbar\left(n + \frac{1}{2}\right), n = 0, 1, 2 \ldots \quad (1)$$

In general, Eq. (1) yields moderately accurate eigenvalues as analytic functions of the parameters contained in the potential.

To properly use the WKB approximation for three-dimensional problems with spherical symmetry is to apply the one-dimensional WKB formalism to the radial Schrodinger equation

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{2m}{\hbar^2}\left[E - V_{eff}(r)\right]\Psi = 0 \quad (2)$$

where the effective potential $V_{eff}(r) = V(r) + \frac{l(l+1)\hbar^2}{2mr^2}$

Such a straightforward application leads to an important difficulty in obtaining exact energy eigenvalue solution because the WKB reduced radial wave function at the origin has a behavior which is different from that of the true wave function[11]. For this reason, Langer[12] suggested that the strength of the angular momentum $l(l + 1)$ should be treated as an adjustable parameter K, not as a fixed quantity. Langer pointed out that K should be replaced with the term $\left(l+\frac{1}{2}\right)^2$ in the lowest order quantization formula which has great physical meaning. The replacement of $l(l+1) \rightarrow \left(l+\frac{1}{2}\right)^2$ regularizes the radial WKB wave function at the origin and ensure correct asymptotic behaviour at large quantum numbers[9-17].

In this work, our aim is to solve the Schrodinger equation for the Manning-Rosen plus Hellmann potential via the WKB approximation method. The Manning-Rosen plus Hellmann potential takes the form:

$$V(r) = -\left[\frac{Ce^{-\alpha r}+De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] - \frac{V_0}{r} + \frac{V_1 e^{-\alpha r}}{r} \tag{3}$$

where α is the screening parameter, C, D and V_0, V_1 are the depth of the potential. Not much has been done in solving the Manning-Rosen Plus Hellmann potential via the WKB method.

This paper is organized as follows: Section 1 has the introduction, a brief description of the semiclassical quantization and the WKB approximation for the radial solution is reviewed in section 2. In section 3, the radial Schrodinger equation with Manning-Rosen plus Hellmann potential is solved. Finally, we give a brief discussion in section 4 before the conclusion in section 5

2. SEMICLASSICAL QUANTIZATION AND THE WKB APPROXIMATION

In this section, we consider the quasiclassical solution of the Schrodinger's equation for the spherically symmetric potentials. Given the Schrodinger equation for a spherically symmetric potentials $V(r)$ of eq. (3) as

$$(-i\hbar)^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right)\psi(r,\theta,\phi) = \left[2m\left(E - V(r)\right)\right] \psi(r,\theta,\phi) \tag{4}$$

The total wave function in Eq. (3) can be defined as

$$\psi(r,\theta,\phi) = [rR(r)]\left[\sqrt{sin\theta}\,\Theta(\theta)\Phi(\phi)\right] \tag{5}$$

And by decomposing the spherical wave function in Eq. (4) using Eq. (5) we obtain the following equations:

$$\left(-i\hbar\frac{d}{dr}\right)^2 R(r) = \left[2m\left(E - V(r)\right) - \frac{\vec{M}^2}{r^2}\right] R(r), \tag{6}$$

$$\left(-i\hbar\frac{d}{d\theta}\right)^2 \Theta(\theta) = \left[\vec{M}^2 - \frac{M_z^2}{\sin^2\theta}\right] \Theta(\theta), \tag{7}$$

$$\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}\phi}\right)^2 \Phi(\phi) = M_z^2\Phi(\phi) \tag{8}$$

where $\vec{M}^2$, M_z^2 are the constants of separation and, at the same time, integrals of motion. The squared angular momentum $\vec{M}^2 = \left(l+\frac{1}{2}\right)^2 \hbar^2$.

Considering Eq. (6), the leading order WKB quantization condition appropriate to Eq. (3) is

$$\int_{r_1}^{r_2} \sqrt{P^2(r)}\, dr = \pi\hbar\left(n+\frac{1}{2}\right), \text{ n=0, 1, 2 . . .} \tag{9}$$

where r_2 & r_1 are the classical turning point known as the roots of the equation

$$P^2(r) = 2m\big(E - V(r)\big) - \frac{\left(l+\frac{1}{2}\right)^2\hbar^2}{r^2} = 0 \tag{10}$$

eq. (9) is the WKB quantization condition which is subject for discussion in the preceding section. Consider Eq. (6)-(8) in the framework of the quasiclassical method, the solution of each of these equations in the leading $\hbar$ approximation can be written in the form

$$\Psi^{WKB}(r) = \frac{A}{\sqrt{P(r,\lambda)}} exp\left[\pm\frac{i}{\hbar}\int \sqrt{P^2(r)}\, dr\right] \tag{11}$$

2.1 Solutions to the radial Schrödinger equation

The radial Schrodinger equation for the Manning-Rosen Plus Hellmann potential can be solved approximately using the WKB quantization condition Eq. (9). Since the potential of interest slowly varies, we assume that the wave function remains sinusoidal. Hence, we use the effective potential and plug it into the WKB approximation of Eq. (10) and to obtain the exact solution, we consider two turning points.

given the effective potential of the centrifugal term as

$$V_{eff}(r) = -\left[\frac{Ce^{-\propto r}+De^{-2\propto r}}{(1-e^{-\propto r})^2}\right] - \frac{V_0}{r} + \frac{V_1 e^{-\propto r}}{r} + \frac{\left(l+\frac{1}{2}\right)^2\hbar^2}{2mr^2} \tag{12}$$

The wave equation (12) is not an exactly solvable problem even for $l = 0$ because of the centrifugal barrier term. Therefore, to solve eq. (12) analytically, we use an approximation scheme of the exponential-type proposed by Greene and Aldrich[12,13] to deal with the centrifugal term:

$$\frac{1}{r^2} = \frac{\alpha^2}{(1- e^{-\alpha r})^2} \tag{13}$$

the potential in Eq. (12) can also be written in the form

$$V_{eff}(r) = -\frac{Ce^{-\propto r}}{(1-e^{-\propto r})^2} - \frac{De^{-2\propto r}}{(1-e^{-\propto r})^2} - \frac{V_0 \propto}{1-e^{-\propto r}} + \frac{V_1 \propto e^{-\propto r}}{1-e^{-\propto r}} + \frac{\alpha^2 \hbar^2 \left(l+\frac{1}{2}\right)^2 e^{-\alpha r}}{2m(1- e^{-\alpha r})^2} \tag{14}$$

Subs. Eq. (14) into Eq. (9), we have

$$\int_{r_1}^{r_2} \sqrt{P^2(r)}\, dr =$$

$$\int_{r_1}^{r_2} \sqrt{2m\left(E_{nl} + \frac{Ce^{-\propto r}}{(1-e^{-\propto r})^2} + \frac{De^{-2\propto r}}{(1-e^{-\propto r})^2} + \frac{V_0 \propto}{1-e^{-\propto r}} - \frac{V_1 \propto e^{-\propto r}}{1-e^{-\propto r}} - \frac{\alpha^2 \hbar^2 \left(l+\frac{1}{2}\right)^2 e^{-\alpha r}}{2m(1- e^{-\alpha r})^2}\right)}\, dr = \pi\left(n+\frac{1}{2}\right) \tag{15}$$

Let $$\vec{M}^2 = \frac{\alpha^2 \hbar^2 \left(l+\frac{1}{2}\right)^2}{2m} \tag{16}$$

$$\int_{r_1}^{r_2} \sqrt{2m\left(E_{nl} + \frac{Ce^{-\propto r}}{(1-e^{-\propto r})^2} + \frac{De^{-2\propto r}}{(1-e^{-\propto r})^2} + \frac{V_0 \propto}{1-e^{-\propto r}} - \frac{V_1 \propto e^{-\propto r}}{1-e^{-\propto r}} - \frac{\vec{M}^2}{(1- e^{-\alpha r})^2}\right)}\, dr = \pi\hbar\left(n+\frac{1}{2}\right) \tag{17}$$

making the transformation

$$z = \frac{e^{-\alpha r}}{1- e^{-\alpha r}} \tag{18}$$

We then obtain

$$\frac{-\sqrt{2m}}{\alpha\hbar}\int_{z_1}^{z_2} \frac{1}{z(1+z)} \sqrt{E_{nl} + Cz(1+z) + V_0\alpha(1+z) - V_1\alpha z + Dz^2 - \vec{M}^2(1+2z+z^2)}dz = \pi\left(n+\frac{1}{2}\right) \tag{19}$$

$$\frac{-\sqrt{2m}}{\alpha\hbar}\int_{z_1}^{z_2} \frac{1}{z(1+z)} \sqrt{-\left(\vec{M}^2 - C - D\right)z^2 + \left(C + V_0\alpha - V_1\alpha - 2\vec{M}^2\right)z + E_{nl} + V_0\alpha - \vec{M}^2}dz = \pi\left(n+\frac{1}{2}\right) \tag{20}$$

$$\frac{-\sqrt{2m(\vec{M}^2 - C - D)}}{\alpha\hbar}\int_{z_1}^{z_2} \frac{1}{z(1+z)} \sqrt{-z^2 + \frac{C+V_0\alpha - V_1\alpha - 2\vec{M}^2}{(\vec{M}^2 - C - D)}z + \frac{E_{nl} + V_0\alpha - \vec{M}^2}{(\vec{M}^2 - C - D)}}dz = \pi\left(n+\frac{1}{2}\right) \tag{21}$$

Let $$\frac{C+V_0\alpha-V_1\alpha-2\vec{M}^2}{(\vec{M}^2-C-D)}=b, \text{ and } \frac{E_{nl}+V_0\alpha-\vec{M}^2}{(\vec{M}^2-C-D)}=-c, \tag{22}$$

We then have

$$\frac{-\sqrt{2m(\vec{M}^2-C-D)}}{\alpha\hbar}\int_{z_1}^{z_2}\frac{1}{z(1+z)}\sqrt{-z^2+bz-c}\,dz=\pi\left(n+\frac{1}{2}\right) \tag{23}$$

$$\frac{-\sqrt{2m(\vec{M}^2-C-D)}}{\alpha}\int_{z_1}^{z_2}\frac{1}{z(1+z)}\sqrt{(z-z_1)(z_2-z)}dz=\pi\hbar\left(n+\frac{1}{2}\right) \tag{24}$$

where we obtain the turning points z_2 & z_1 from the terms inside the square roots as

$$z_1=\frac{-b-\sqrt{b^2-4C}}{2}$$

$$z_2=\frac{-b+\sqrt{b-4C}}{2}$$

Let $$2z+1=y; dz=\frac{dy}{2} \tag{25}$$

subs. Eq. (25) into eq. (24), we obtain

$$\int_{y_1}^{y_2}\frac{1}{y^2-1}\sqrt{(y-y_1)(y_2-y)}dy=\frac{-\alpha\pi\hbar\left(n+\frac{1}{2}\right)}{\sqrt{2m(\vec{M}^2-C-D)}} \tag{26}$$

For computing the integral in equation (26), we use the integral expression [13, 14]

$$\int_{y_1}^{y_2}\frac{1}{y^2-1}\sqrt{(y-y_1)(y_2-y)}dy=\frac{\pi}{2}\left[\sqrt{(y_1+1)(y_2+1)}-\sqrt{(y_1-1)(y_2-1)}+2\right] \tag{27}$$

where the limits y_1, y_2 are real numbers, with $y_1 < y_2$. Comparing equation (27) with equation (26), and solving for E_{nl} gives

$$E_{nl}=-\frac{\alpha^2\hbar^2}{2\mu}\left\{\frac{2\mu V_0}{\alpha\hbar^2}-\left(l+\frac{1}{2}\right)^2\left[\frac{2\left(l+\frac{1}{2}\right)^2-\frac{2\mu C}{\alpha^2\hbar^2}+\left(n+\frac{1}{2}\right)^2-\frac{2\mu V_0}{\alpha\hbar^2}+\frac{2\mu V_1}{\alpha\hbar^2}+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu C}{\alpha^2\hbar^2}-\frac{2\mu D}{\alpha^2\hbar^2}}}{2n+1+2\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu D}{\alpha^2\hbar^2}-\frac{2\mu C}{\alpha^2\hbar^2}}}\right]^2\right\} \tag{28}$$

2.2 Discussion

Case 1: If $V_0 = V_1 = 0$ in eq. (3), we obtain the energy equation of Manning-Rosen potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2\left(l+\frac{1}{2}\right)^2-\frac{2\mu C}{\alpha^2\hbar^2}+\left(n+\frac{1}{2}\right)^2+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu C}{\alpha^2\hbar^2}-\frac{2\mu D}{\alpha^2\hbar^2}}}{2n+1+2\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu D}{\alpha^2\hbar^2}-\frac{2\mu C}{\alpha^2\hbar^2}}}\right]^2-\left(l+\frac{1}{2}\right)^2\right\} \tag{29}$$

Case 2: If $C = D = 0$ in eq. (3), we obtain the energy equation of the Hellmann potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\frac{2\mu V_0}{\alpha\hbar^2}-\left(l+\frac{1}{2}\right)^2\left[\frac{2\left(l+\frac{1}{2}\right)^2+\left(n+\frac{1}{2}\right)^2-\frac{2\mu V_0}{\alpha\hbar^2}+\frac{2\mu V_1}{\alpha\hbar^2}+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2}}{2(n+l+1)}\right]^2\right\} \tag{30}$$

Case 3: If $V_1 = 0, C = D = 0$ in eq. (3), we obtain the energy equation of the coulomb potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\frac{2\mu V_0}{\alpha\hbar^2}-\left(l+\frac{1}{2}\right)^2\left[\frac{2\left(l+\frac{1}{2}\right)^2+\left(n+\frac{1}{2}\right)^2-\frac{2\mu V_0}{\alpha\hbar^2}+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2}}{2(n+l+1)}\right]^2\right\} \tag{31}$$

Case 4: If one $V_0 = 0, V_1 = -V_1, C = D = 0$ in eq. (3), we obtain the energy equation of the Yukawa potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2\left(l+\frac{1}{2}\right)^2+\left(n+\frac{1}{2}\right)^2-\frac{2\mu V_1}{\alpha\hbar^2}+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2}}{2(n+l+1)}\right]^2-\left(l+\frac{1}{2}\right)^2\right\} \tag{32}$$

3.0 CONCLUSION

In this paper, we present the approximate energy spectrum for Manning-Rosen plus Hellmann potential using the Wentzel-Kramers-Brillouin WKB approach. The energy eigenvalues and the corresponding total normalized wave functions expressed in terms of the hypergeometric functions for the system are also obtained.

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